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The Characterization of the Quasi-typical Extension of an Inner Product

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In this paper, we generalize a discrete model of the inner product associated with a distribution function, called a typical inner product. In this case, the nth orthogonal polynomial has multiple roots. Necessary and sufficient conditions for the existence of quasi-typical extensions for an inner product defined in $\mathscr{P}_{n-1}$ (vector space of polynomials of degree $\leqslant n-1$) are given. We also obtain a noteworthy expression for the nth moments in terms of the above moments. © 1990 Academic Press, Inc.

1. INTRODUCTION

Let γ be a Jordan arc, and $\sigma_1, \sigma_2, ..., \sigma_h$ distribution functions defined on γ; that is, for $j = 1, 2, ..., h$:

(a) $\sigma_j : \gamma \to R$ is a nondecreasing function in terms of the parametrization of the Jordan arc.

(b) $|\int_\gamma z^k \bar{z}^l \, d\sigma_j| < \infty$, for $k, l = 0, 1, 2, ...$.

We consider $L^2_\gamma(\sigma_1, \sigma_2, ..., \sigma_h)$ defined as

$$\left\{ f : \gamma \to \mathbb{C} \;\middle|\; \sum_{i=1}^{h} \int |f^{(i-1)}(z)|^2 \, d\sigma_i < \infty \right\},$$

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with the usual inner product

$$\langle f, g\rangle = \sum_{i=1}^{h} \int_{\gamma} f^{(i-1)}(z)\, \overline{g^{(i-1)}(z)}\, d\sigma_i. \tag{1}$$

For a compact interval A of the real line, several properties of orthogonal polynomials associated with this inner product have been studied in [2]. For example, the zeros of these polynomials need not be distinct or lie in A. There are no references for similar problems in the theory of orthogonal polynomials over curves or arcs. In this paper, we analyze a discretization of (1) and compare it with the typical inner product which appears in the theory of moments and Gaussian quadratures (see [1, 4]).

Let $\alpha_1, \alpha_2, ..., \alpha_k \in \gamma$, and let $\sigma_i : \gamma \to \mathbb{R}$ $(i = 1, 2, ..., h)$ be defined by $\sigma_i(z) = K_{ji}$ if $\alpha_{j-1} < z < \alpha_j$, with the natural orientation on the arc γ induced by the parametrization (see [3]), where the K_{ji} satisfy $p_{ji} = K_{j+1,i} - K_{ji} > 0$ for each $j = 1, 2, ..., k-1$ and $i = 1, 2, ..., h$. Moreover, if $p_{ji} = 0$, we assume $p_{jl} = 0$ for each $l > i$.

This discretization of (1) allows us to represent it as

$$\langle f, g\rangle = \sum_{j=1}^{k} \sum_{m=0}^{\mu_j - 1} f^{(m)}(\alpha_j)\, \overline{g^{(m)}(\alpha_j)}\, p_{j,m+1}, \tag{2}$$

where μ_j $(1 \leqslant \mu_j \leqslant h)$ is the number of distribution functions σ_i which have a jump at α_j, that is, $p_{ij} > 0$ for $i = 1, ..., \mu_j$.

Remarks. 1. If $\sigma_i = \sigma_1$ for each i, (1) is the inner product which defines the Sobolev spaces.

2. If $\mu_j = 1$, $j = 1, 2, ..., k$, we have a degenerate situation studied in [3, 5] for some particular classes of algebraic curves, namely, harmonic algebraic curves and equipotential curves.

3. $\tilde{P}_n(z) = (z - \alpha_1)^{\mu_1} (z - \alpha_2)^{\mu_2} \cdots (z - \alpha_k)^{\mu_k}$ is the nth monic orthogonal polynomial for the inner product (2), where $\sum_{j=1}^{k} \mu_j = n$.

2. Quasi-typical Inner Product

2.1. Definition. Let $\mu_1, \mu_2, ..., \mu_k$ be positive integers, let $\sum_{i=1}^{k} \mu_i = n$, let $\alpha_1, \alpha_2, ..., \alpha_k$, $1 \leqslant k \leqslant n$, be complex numbers, and let $p_{11}, ..., p_{1\mu_1}, ..., p_{k1}, ..., p_{k\mu_k}$ be positive real numbers.

A quasi-typical inner product is an inner product defined in $\mathscr{P}_n$ by the Gram matrix $M_n = (c_{ij})_{i,j=0}^n$, where

$$c_{lh} = \langle z^l, z^h \rangle = \sum_{i=1}^{k} \sum_{m=0}^{\mu_i - 1} V_{lm} V_{hm} \alpha_i^{l-m} \bar{\alpha}_i^{h-m} p_{i,m+1}$$

$(l, h = 0, 1, ..., n;\ (l, h) \neq (n, n))$,

$$c_{nn} = \sum_{i=1}^{k} \sum_{m=0}^{\mu_i - 1} (V_{nm})^2 \alpha_i^{n-m} \bar{\alpha}_i^{n-m} p_{i,m+1} + e_n,$$

$e_n > 0$ and $V_{nm} = n(n-1) \cdots (n-m+1)$; $n, m = 0, 1, 2, ...$.

The Gram matrix M_{n-1}, also called the generalized moment matrix, is congruent to a nonnegative diagonal matrix P,

$$M_{n-1} = (V'_{n-1})^* P V'_{n-1},$$

where

$$P = \begin{pmatrix} p_{11} & & & & & \\ & \ddots & & & \bigcirc & \\ & & p_{1\mu_1} & & & \\ & & & \ddots & & \\ & & & & p_{k1} & \\ & \bigcirc & & & & \ddots \\ & & & & & & p_{k\mu_k} \end{pmatrix}$$

and V'_{n-1} is the generalized Van der Monde matrix.

If $k = n$, we obtain the so-called typical inner product, which has been studied in [3, 5] for some canonical types of algebraic curves in the complex plane.

2.2. Proposition. *For a quasi-typical inner product defined in* $\mathscr{P}_n$:

(a) *If* $Q(z)$, $R(z) \in \mathscr{P}_n$ *and the degree of* $Q(z)$ *or the degree of* $R(z)$ *is* $\neq n$, *then*

$$\langle Q(z), R(z) \rangle = \sum_{i=1}^{k} \sum_{m=0}^{\mu_i - 1} p_{i,m+1} Q^{(m)}(\alpha_i) \overline{R^{(m)}(\alpha_i)}.$$

(b) *If* $Q(z)$ *and* $R(z) \in \mathscr{P}_n$ *and their degree is exactly* n, *then*

$$\langle Q(z), R(z) \rangle = \sum_{i=1}^{k} \sum_{m=0}^{\mu_i - 1} p_{i,m+1} Q^{(m)}(\alpha_i) \overline{R^{(m)}(\alpha_i)} + q_n \bar{r}_n e_n,$$

where q_n, r_n *are the leading coefficients of* $Q(z)$ *and* $R(z)$, *respectively.*

(c) *The zeros of the nth monic orthogonal polynomial* $\tilde{P}_n(z)$ *are* $\alpha_1, \alpha_2, ..., \alpha_k$, *with multiplicities* $\mu_1, \mu_2, ..., \mu_k$, *respectively.*

Proof. (a) and (b) are obvious by using linearity properties.
(c) For each $Q(z) \in \mathcal{P}_{n-1}$,

$$0 = \langle \tilde{P}_n(z), Q(z) \rangle = \sum_{i=1}^{k} \sum_{m=0}^{\mu_i - 1} p_{i,m+1} \tilde{P}_n^{(m)}(\alpha_i) \overline{Q^{(m)}(\alpha_i)}.$$

If we consider the Lagrange–Sylvester polynomials $L_{i,m}(z)$, $i = 1, ..., k$; $m = 0, ..., \mu_i - 1$, which are characterized by

$$L_{i,m}^{(s)}(\alpha_t) = \delta_{sm} \delta_{it},$$

we get $p_{i,m+1} \tilde{P}_n^{(m)}(\alpha_i) = 0$, and the result follows immediately. ∎

3. Quasi-typical Extensions

3.1. Definition. Let $m_{n-1} = (c_{ij})_{i,j=0}^{n-1}$ be a positive definite Hermitian matrix.

A matrix $m_n = (d_{ij})_{i,j=0}^{n}$ is a quasi-typical extension of the matrix m_{n-1} if

(a) There exist complex numbers $\alpha_1, \alpha_2, ..., \alpha_k$ and positive real numbers $p_{11}, ..., p_{1\mu_1}, ..., p_{k1}, ..., p_{k\mu_k}, e_n$, with $\sum_{i=1}^{k} \mu_i = n$ such that

$$d_{lh} = \sum_{i=1}^{k} \sum_{m=0}^{\mu_i - 1} V_{lm} V_{hm} p_{i,m+1} \alpha_i^{l-m} \bar{\alpha}_i^{h-m},$$

$l, h = 0, ..., n$, $(l, h) \neq (n, n)$;

$$d_{nn} = \sum_{i=1}^{k} \sum_{m=0}^{\mu_i - 1} (V_{nm})^2 \alpha_i^{n-m} \bar{\alpha}_i^{n-m} p_{i,m+1} + e_n;$$

(b) $d_{lh} = c_{lh}$, for $l, h = 0, ..., n-1$.

3.2. Proposition. *Let* $m_{n-1} = (c_{ij})_{i,j=0}^{n-1}$ *the Gram matrix of an inner product defined in* $\mathcal{P}_{n-1}$.
The following properties are equivalent:

(i) *A quasi-typical extension for* m_{n-1} *exists.*

(ii) $\{L_{i,m}(z) \mid i = 1, ..., k, m = 0, ..., \mu_i - 1\}$ *is an orthogonal basis in* $\mathcal{P}_{n-1}$.

(iii) $\{(\partial^l K_{n-1}(z, \alpha_i)/\partial\bar{\alpha}_i^l) \mid i=1, ..., k;\ l=0, ..., \mu_i - 1\}$ *is an orthogonal basis in* $\mathscr{P}_{n-1}$, *where we denote*

$$\frac{\partial^l K_{n-1}(z, \alpha_i)}{\partial\bar{\alpha}_i^l} = \left[\overline{\frac{\partial^l K_{n-1}(y, z)}{\partial y^l}}\right]_{y=\alpha_i},$$

and where $K_{n-1}(z, y)$ *is the* $(n-1)$ *st kernel*

$$K_{n-1}(z, y) = \sum_{j=0}^{n-1} \frac{1}{e_j} \tilde{P}_j(z)\, \overline{\tilde{P}_j(y)}; \qquad e_j = \frac{\det m_j}{\det m_{j-1}},$$

for $j=1, ..., n$, *and* $e_0 = c_{00}$.

Proof. (i) $\Rightarrow$ (ii).

$$\langle L_{j,s}(z), L_{l,t}(z)\rangle = \sum_{i=1}^{k} \sum_{m=0}^{\mu_i - 1} p_{i,m+1} L_{j,s}^{(m)}(\alpha_i)\, \overline{L_{l,t}^{(m)}(\alpha_i)}$$
$$= p_{j,s+1}\delta_{jl}\delta_{st}.$$

Then $\{L_{j,s}(z)\}$ constitutes an orthogonal system in $\mathscr{P}_{n-1}$ and, consequently, a basis.

(ii) $\Rightarrow$ (iii).

$$\frac{\partial^j K_{n-1}(z, \alpha_t)}{\partial\bar{\alpha}_t^j} = \sum_{i=1}^{k} \sum_{m=0}^{\mu_i - 1} a_{tj}^{(i,m)} L_{i,m}(z).$$

Then

$$a_{t,j}^{(i,m)}\langle L_{i,m}(z), L_{i,m}(z)\rangle = \left\langle \frac{\partial^j K_{n-1}(z, \alpha_t)}{\partial\bar{\alpha}_t^j}, L_{i,m}(z)\right\rangle$$
$$= \overline{L_{i,m}^{(j)}(\alpha_t)} = \delta_{jm}\delta_{it},$$

and

$$\frac{\partial^j K_{n-1}(z, \alpha_t)}{\partial\bar{\alpha}_t^j} = \frac{L_{t,j}(z)}{\langle L_{t,j}(z), L_{t,j}(z)\rangle} = \lambda_{tj} L_{t,j}(z).$$

(iii) $\Rightarrow$ (i).

$$L_{i,m}(z) = \sum_{t=1}^{k} \sum_{j=0}^{\mu_i - 1} b_{im}^{(t,j)} \frac{\partial^j K_{n-1}(z, \alpha_t)}{\partial\bar{\alpha}_t^j}.$$

Therefore

$$b_{im}^{(t,j)} \left\langle \frac{\partial^j K_{n-1}(z, \alpha_t)}{\partial \bar{\alpha}_t^j}, \frac{\partial^j K_{n-1}(z, \alpha_t)}{\partial \bar{\alpha}_t^j} \right\rangle$$

$$= \left\langle L_{i,m}(z), \frac{\partial^j K_{n-1}(z, \alpha_t)}{\partial \bar{\alpha}_t^j} \right\rangle = L_{im}^{(j)}(\alpha_t) = \delta_{jm} \delta_{it},$$

and

$$L_{i,m}(z) = \left[\sum_{h=0}^{n-1} \frac{1}{e_h} \tilde{P}_h^{(m)}(\alpha_i) \overline{\tilde{P}_h^{(m)}(\alpha_i)} \right]^{-1} \frac{\partial^m K_{n-1}(z, \alpha_i)}{\partial \bar{\alpha}_i^m}. \tag{3}$$

Since

$$z^l = \sum_{i=1}^{k} \sum_{m=0}^{\mu_i - 1} V_{lm} \alpha_i^{l-m} L_{i,m}(z) \qquad (l = 0, ..., n-1),$$

we have

$$\langle z^l, z^h \rangle = \left\langle \sum_{i=1}^{k} \sum_{m=0}^{\mu_i - 1} V_{lm} \alpha_i^{l-m} L_{i,m}(z), \sum_{j=1}^{k} \sum_{s=0}^{\mu_j - 1} V_{hs} \alpha_j^{h-s} L_{j,s}(z) \right\rangle$$

$$= \sum_{i=1}^{k} \sum_{m=0}^{\mu_i - 1} V_{lm} V_{hm} \alpha_i^{l-m} \bar{\alpha}_i^{h-m} \langle L_{i,m}(z), L_{i,m}(z) \rangle$$

$$= \sum_{i=1}^{k} \sum_{m=0}^{\mu_i - 1} V_{lm} V_{hm} \alpha_i^{l-m} \bar{\alpha}_i^{h-m} p_{i,m+1},$$

where

$$p_{i,m+1} = \left[\sum_{r=0}^{n-1} \frac{1}{e_r} \tilde{P}_r^{(m)}(\alpha_i) \overline{\tilde{P}_r^{(m)}(\alpha_i)} \right]^{-1} > 0.$$

Then we can define the quasi-typical extension as usual. ∎

As a very simple consequence of (3), we have

$$(\mu_i - 1)! \frac{\overline{\tilde{P}_{n-1}^{(\mu_i - 1)}(\alpha_i)}}{e_{n-1}} = \sum_{h=0}^{n-1} \frac{1}{e_h} \tilde{P}_h^{(\mu_i - 1)}(\alpha_i) \overline{\tilde{P}_h^{(\mu_i - 1)}(\alpha_i)} \prod_{\substack{j=1 \\ j \neq i}}^{k} (\alpha_i - \alpha_j)^{-\mu_j}. \tag{4}$$

Finally, we obtain a remarkable expression for the nth monic orthogonal polynomial associated with the quasi-typical extension in terms of the above mentioned orthogonal basis.

3.3. PROPOSITION. *For a quasi-typical extension of an inner product defined in* $\mathscr{P}_{n-1}$,

$$\text{(i)}\quad \tilde{P}_n(z) = \frac{e_{n-1}}{\tilde{P}_{n-1}^{(\mu_i-1)}(\alpha_i)} \frac{\partial^{\mu_i-1} K_{n-1}(z,\alpha_i)}{\partial \bar{\alpha}_i^{\mu_i-1}} (z-\alpha_i)$$

$\tilde{P}_{n-1}^{(\mu_i-1)}(\alpha_i) \neq 0$, $i=1, ..., k$, by (4)).

$$\text{(ii)}\quad c_{nh} = \frac{1}{\det m_{n-2} \tilde{P}_{n-1}^{(\mu_i-1)}(\alpha_i)} \times \begin{vmatrix} c_{00} & \cdots & c_{n-2,0} & c_{n-1,0} & (1)^{(\mu_i-1)} \\ c_{01} & \cdots & c_{n-2,1} & c_{n-1,1} & \bar{\alpha}_i^{(\mu_i-1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{0,n-1} & \cdots & c_{n-2,n-1} & c_{n-1,n-1} & (\overline{\alpha_i^{n-1}})^{(\mu_i-1)} \\ c_{1h} & \cdots & c_{n-1,h} & 0 & \alpha_i(\overline{\alpha_i^h})^{(\mu_i-1)} \end{vmatrix}$$

($h=0, ..., n-1$), *where*

$$(\overline{\alpha_i^j})^{(\mu_j-1)} = V_{j,\mu_j-1} \bar{\alpha}_i^{j-\mu_i+1},$$

and $V_{j,\mu_i-1}=0$ *if* $j<\mu_i-1$.

Proof. (i) is an immediate consequence of (3) and (4).

(ii) By the above expression, for $h=0, ..., n-1$,

$$0 = \left\langle (z-\alpha_i) \frac{\partial^{\mu_i-1} K_{n-1}(z,\alpha_i)}{\partial \bar{\alpha}_i^{\mu_i-1}}, z^h \right\rangle \Rightarrow$$

$$\alpha_i \left\langle \frac{\partial^{\mu_i-1} K_{n-1}(z,\alpha_i)}{\partial \bar{\alpha}_i^{\mu_i-1}}, z^h \right\rangle = \alpha_i V_{h,\mu_i-1} \bar{\alpha}_i^{h-\mu_i+1}.$$

But, since

$$K_{n-1}(z,\alpha_i) = -\frac{1}{\Delta_{n-1}} \begin{vmatrix} c_{00} & c_{10} & \cdots & c_{n-1,0} & 1 \\ c_{01} & c_{11} & \cdots & c_{n-1,1} & \bar{\alpha}_i \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{0,n-1} & c_{1,n-1} & \cdots & c_{n-1,n-1} & \bar{\alpha}_i^{n-1} \\ 1 & z & \cdots & z^{n-1} & 0 \end{vmatrix},$$

$$\left\langle z \frac{\partial^{\mu_i-1} K_{n-1}(z,\alpha_i)}{\partial \bar{\alpha}_i^{\mu_i-1}}, z^h \right\rangle = -\frac{1}{\Delta_{n-1}} \begin{vmatrix} c_{00} & \cdots & c_{n-1,0} & (1)^{(\mu_i-1)} \\ \cdots & \cdots & \cdots & \cdots \\ c_{0,n-1} & \cdots & c_{n-1,n-1} & (\overline{\alpha_i^{n-1}})^{(\mu_i-1)} \\ c_{1h} & \cdots & c_{nh} & 0 \end{vmatrix}$$

$$= \alpha_i \overline{(\alpha_i^h)}^{(\mu_i-1)}.$$

Then,

$$\left|\begin{array}{c|c} & (1)^{(\mu_i - 1)} \\ & \vdots \\ m_{n-1} & \overline{(\alpha_i^{n-1})}^{(\mu_i - 1)} \\ \hline c_{1h} \cdots c_{nh} & \alpha_i \overline{(\alpha_i^{h})}^{(\mu_i - 1)} \end{array}\right| = 0,$$

and

$$c_{nh} \left|\begin{array}{c|c} & (1)^{(\mu_i - 1)} \\ m_{n-2} & \vdots \\ \hline c_{0,n-1} \cdots c_{n-2,n-1} & \overline{(\alpha_i^{n-1})}^{(\mu_i - 1)} \end{array}\right|$$

$$= \left|\begin{array}{c|c} & (1)^{(\mu_i - 1)} \\ & \vdots \\ m_{n-1} & \overline{(\alpha_i^{n-1})}^{(\mu_i - 1)} \\ \hline c_{1h} \cdots c_{n-1,h} \; 0 & \alpha_i \overline{(\alpha_i^{h})}^{(\mu_i - 1)} \end{array}\right|.$$

Now (ii) follows immediately. ∎

Equality (ii) allows us to obtain the moment of order (n, h) in terms of the above moments.

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